Lensing Effect of a Cosmic String in Chern-Simons Gravity

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It is pointed out that any conformally transformed of a flat space-time metric $\mathring{g}_{ij} = f(x)\ \eta_{ij}$ is a solution to Witten's equation of Chern-Simons gravity, which holds outside matter in (2+1) dimensions. It is also shown that a simultaneous exterior solution to both Witten's and Einstein's equations, yields the lensing effect of an isolated cosmic string, if $f(x)$ is reduced to an arbitrary dimensionless constant. However, the solution to Witten's equation with $f(x)$ being an arbitrary and continuous function of space-time coordinates, also leads to an open circular path for a light ray near the string.

Chern-Simons gravity (CSG) is a (2+1)-dimensional gauge model for gravity [1,2], which is in essence different from general relativity (GR). Such an approach has been formulated as a field theory in a three-dimensional curved space-time, and it contrasts with GR which states that in three space-time dimensions space-time is flat outside matter. Indeed, GR in this case has some characteristic features that distinguish it from the more familiar (3+1)-dimensional gravity [3], since in (2+1)-dimensions GR field equations lead to a flat space-time outside matter with a conical symmetry, where Einstein's theory accomplishes for topological defects.

Otherwise, it has been shown [4] that (2+1)-dimensional gravity is equivalent to a gauge theory, with a pure Chern-Simons action, and a gauge group $ISO(2,1), SO(3,1)$, or $SO(2,2)$ respectively, depending on the value of the cosmological constant. We recall that when a cosmological constant is included in (2+1)-dimensional GR, Minkowski space-time is replaced by either a de Sitter space or an anti-de Sitter space, and $ISO(2,1)$ is replaced by $SO(3,1)$ or $SO(2,2)$, respectively. In the present paper we derive possible physical solutions to the field equation of Chern-Simons gravity, within a particle dynamics framework, and

afterwards we study the trajectory of photons in the neighbourhood of an isolated cosmic string.

As has been pointed out [2], a gauge model for (2+1)-dimensional gravity for the $SO(3,2)$ group has a Chern-Simons interpretation, where the fundamental variable is the vielbein e_i^a and the spin connection w_i^a is a function of e_i^a, by requiring that $D_i e_j^a - D_j e_i^a = 0$, ($i, j, k$ are world indices and a, b, c, are Lorentz indices). The topological Chern-Simons action in such a model is constructed from the Riemann curvature tensor, which results to be

$$I = \int_{\mathcal{M}} \epsilon^{ijk} \left[\omega_{ia} (\partial_j \omega_k^a - \partial_k \omega_j^a) + \frac{2}{3} \epsilon_{abc} \omega_i^a \omega_j^b \omega_k^c \right] \tag{1}$$

If we vary Eq.(1) with respect to e_i^a, we obtain the field equation outside matter

$$D_k W_{ij} - D_j W_{ik} = 0 \tag{2}$$

where $W_{ij} = R_{ij} - (1/4) g_{ij} R$. Notice that Eq.(2) is a conformally invariant equation, since its left-hand side is the three-dimensional analogue of the Weyl tensor. Indeed, the vanishing of Eq.(2) is the condition asserting that a (2+1)-dimensional space-time outside matter is a conformally flat manifold in the CSG model.

Recall that the conformal group is the group of diffeomorphisms of compactified Minkowski space that leaves the space-time metric invariant up to a Weyl rescaling $\mathring{g}_{ij} \rightarrow f(x)\ \eta_{ij}$, where $f(x)$ is an arbitrary and continuous function of space-time coordinates. In (2+1)-dimensions the conformal group has ten generators and it is isomorphic to SO(3,2), in the case of a Lorentzian signature $(-, +, +)$. Since Eq.(2) states that (2+1)-dimensional space-time is conformally flat, then any conformally flat space-time metric given in the form $ds^2 = f(t, r, \theta)\ (-c^2 dt^2 + dr^2 + r^2 d\theta^2)$ should be a solution to that equation. Notice that the metric tensor has in this case only the diagonal components non-vanishing *i. e.* $\mathring{g}_{00} = -\mathring{g}_{11} = -f(t, r, \theta)$, $\mathring{g}_{22} = f(t, r, \theta)\ r^2$, and an easy computation shows that $D^i \mathring{g}_{ij} = 0$, which means that ω_i^a is a metric connection. Moreover, the W_{ij} thus obtained satisfy Eq.(2). The above metric form can be written as

$$ds^2 = -f(t, r, \theta)\ \gamma^{-2}\ c^2\ dt^2 \tag{3}$$

where $\gamma = (1-\beta^2)^{-1/2} = (1-v^2/c^2)^{-1/2}$ and v is the velocity of a particle in this field, as defined in terms of the coordinate time t.

With ds^2 given in Eq.(3) the relativistic action for a particle with rest mass m under the influence of gravity is $S = imc \int ds = \int Ldt$, where $L = -mc^2\sqrt{f(t,r,\theta)(1-\beta^2)}$ is the Lagrangian of this particle. The above Lagrangian turns into the Lagrangian of special relativity for a free particle in a vanishing field approximation $f(t,r,\theta) \to 1$.

Now we derive the relativistic particle dynamics within the present framework, and afterwards we consider the motion of this particle in the neighbourhood of an isolated cosmic string. Taking into account the Lagrangian of the particle already obtained, we see that the spatial components of its relativistic momentum are

$$p^\alpha = \frac{\partial L}{\partial v_\alpha} = \sqrt{f(x)}\,\gamma\, m\, v^\alpha, \qquad (\alpha = 1,2) \tag{4}$$

where v^α is the corresponding component of the particle's velocity. The above result suggests that gravity emerges in the CSG framework through a non-minimal coupling.

Under a Legendre transformation of L, the relativistic Hamiltonian of the particle is

$$H = \sum_{\alpha=1}^{2} p_\alpha v^\alpha - L = \sqrt{f(x)}\,\gamma\, m\, c^2 \tag{5}$$

which represents the energy E of the particle properly, in a conservative case. Indeed, this is the relativistic quantity that is conserved during the motion of the particle under the influence of gravity in the CSG approach.

In polar coordinates the particle's Lagrangian reads

$$L = -\sqrt{f(x)}\left(1 - \frac{\dot{r}^2}{c^2} - \frac{r^2\dot{\theta}^2}{c^2}\right)^{\frac{1}{2}} m\, c^2 \tag{6}$$

where the dot denotes derivative with respect to the time coordinate. The canonical momenta are then

$$P_r = \frac{\partial L}{\partial \dot{r}} = \sqrt{f(x)}\,\gamma\, m\, \dot{r}, \qquad P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \sqrt{f(x)}\,\gamma\, N = \Lambda \tag{7}$$

Here Λ plays the role of a relativistic angular momentum of the particle in the presence of gravity and $N = mr^2\dot{\theta}$ is its Newtonian angular momentum. If we write Euler-Lagrange

equations with the Lagrangian (6), we conclude that the canonical momentum Λ is a constant of motion. Notice that, although the Newtonian angular momentum is still conserved in this case, the quantity we must deal with is Λ, instead of N.

With the above results the Hamiltonian of the particle becomes

$$H = P_r \dot{r} + P_\theta \dot{\theta} - L = \frac{c^2 P_r^2}{E} + \frac{c^2 \Lambda^2}{E r^2} + \frac{m^2\, c^4 f(x)}{E} \tag{8}$$

where we have considered the energy given in Eq.(5). Since the action and the Hamiltonian are related by $H + \partial S/\partial t = 0$ for a conservative case, then we can write S in the form

$$S = -E\, t + F(r, \Lambda, E) + \Lambda\theta \tag{9}$$

where F is an unknown function of r and of the constants of motion. Otherwise, since $P_r = \partial S/\partial r = dF/dr$ and $P_\theta = \partial S/\partial\theta = \Lambda$, then Eq.(8) turns into

$$\frac{c^2}{E}\left(\frac{dF}{dr}\right)^2 + \frac{c^2\Lambda^2}{Er^2} + \frac{m^2 c^4 f(x)}{E} - E = 0 \tag{10}$$

Now, if we integrate Eq.(10) we obtain for the action (9)

$$S = \int \sqrt{\frac{E^2}{c^2} - f(x) m^2 c^2 - \Lambda^2/r^2}\ dr + \Lambda\theta - Et \tag{11}$$

The path of a particle is derived from the condition $\partial S/\partial\Lambda = 0$, which yields

$$\theta = \int \frac{\Lambda\ dr}{r^2\sqrt{E^2/c^2 - f(x)m^2c^2 - \Lambda^2/r^2}} \tag{12}$$

however, from Eqs.(4) and (5) we easily conclude that

$$E = \sqrt{p^2\ c^2 + f(x) m^2 c^4} \tag{13}$$

which states that for photons *i. e.* $m = 0$ we still have $E = p\ c$ as it happens in special relativity. Thus, according to Eqs.(12) and (13), and taking into account that in the case of photons we can assume $\Lambda = b\, p$, where b is the impact parameter, we finally obtain

$$\theta = \int \frac{dr}{r^2\sqrt{1/b^2 - 1/r^2}} \tag{14}$$

Next step is to obtain a simultaneous solution to both Witten's equation (2) and to Einstein's equation outside matter. For that we assume a static case with axial symmetry, where $f(x) = f(r)$. Thus, the only non-vanishing components of the Riemannian connection derived from Eq.(3) are

$$\Gamma^0_{01} = \Gamma^0_{10} = \Gamma^1_{00} = \Gamma^1_{11} = \frac{f'(r)}{2f(r)}, \quad \Gamma^1_{22} = -\frac{r^2 f'(r) + 2rf(r)}{2f(r)},$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{2f(r) + rf'(r)}{2rf(r)} \tag{15}$$

where the prime denotes differentiation with respect to r. The only nonzero components of the Ricci tensor are in this case

$$R_{00} = \frac{2f(r)f'(r) - r[f'(r)]^2 + 2rf(r)f''(r)}{4r[f(r)]^2}, \qquad R_{11} = \frac{[f'(r)]^2}{[f(r)]^2} - \frac{f'(r)}{2rf(r)} - \frac{f''(r)}{f(r)} \tag{16}$$

$$R_{22} = \frac{r^2[f'(r)]^2 - 4rf(r)f'(r) - 2r^2 f(r)f''(r)}{4[f(r)]^2}$$

and the scalar curvature outside matter is

$$R = \frac{3r[f'(r)]^2 - 4f(r)f'(r) - 4rf(r)f''(r)}{2r[f(r)]^3} \tag{17}$$

Hence, the only nonzero components of the Einstein tensor outside matter are

$$G_{00} = \frac{r[f'(r)]^2 - rf(r)f''(r) - f(r)f'(r)}{2r[f(r)]^2}$$

$$G_{11} = \frac{2f'(r)f(r) + r[f'(r)]^2}{4r[f(r)]^2} \tag{18}$$

$$G_{22} = \frac{r^2 f(r)f''(r) - r^2[f'(r)]^2}{2[f(r)]^2}$$

and the condition $G_{ij} = 0$ leads to the differential equations

$$f'(r)[rf'(r) - f(r)] - rf(r)f''(r) = 0, \qquad f'(r)[2f(r) + rf'(r)] = 0,$$

$$f(r)f''(r) - [f'(r)]^2 = 0 \tag{19}$$

The second equation above has the solutions $f'(r) = 0$ and $f(r) = A/r^2$, where A is an arbitrary constant, however, the first and the third equations above are satisfied if and only if $f'(r) = 0$ *i. e.* $f(r) = B =$ constant. Hence the solution to Eqs.(19) is $f = B$.

Otherwise, Einstein's field equation inside matter $R_{ij} - g_{ij}R/2 = (8\pi\kappa/c^2)\ T_{ij}$, ($\kappa$ is Newtons's constant) must be considered with appropriate values of T_{ij} for a cosmic string in the interior solution. In this case we assume the metric in the form

$$ds^2 = -c^2dt^2 + a^2d\theta^2 + a^2\sin^2\theta d\phi^2 + dz^2 \tag{20}$$

where $-\infty \le t \le +\infty,\ 0 \le \theta \le \theta_M,\ 0 \le \phi \le 2\pi,\ -\infty \le z \le +\infty$ and a is the radius of curvature of a spherical cap inside the string [5]. For the above metric the only nonzero connection coefficients are $\Gamma^1_{22} = -\sin\theta\cos\theta$ and $\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$ and the only nonzero components of the Ricci tensor are $R_{11} = 1,\ R_{22} = \sin^2\theta$. Thus, the scalar curvature inside the string is $R = 2/a^2$ and, in consequence of Einstein's equation inside matter, the only nonzero components of the energy momentum tensor are $T_{00} = -T_{33} = -c^2/(8\pi\kappa a^2) = -\rho$ where ρ is the volumetric density of mass. These results lead to a deficit angle produced by the string $\Delta = 2\pi\ (1 - \cos\theta_M) = 8\pi\kappa\mu/c^2$, where μ is the linear mass density.

With the exact and unique form $f(x) = B$ before derived, the space-time metric (3) outside the string becomes

$$ds^2 = B\ (-c^2dt^2 + dr^2 + r^2d\theta^2) \tag{21}$$

however, we can define the new coordinates $t' = \sqrt{B}\ t$ and $r' = \sqrt{B}\ r$, which yield the usual form

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2, \qquad 0 \le \theta \le \theta_M \tag{22}$$

after dropping the primes. Notice that with the space-time metric given in the above form, and with the assumption $f(x) = B$, the CSG framework before derived [Eqs.(4) - (12)] is reduced to a free particle approach, as it happens in GR.

Let us now return to Eq.(14). Its integration yields $1/r = b^{-1} \cos\theta$, stating that photons should follow straight lines near a cosmic string. However, such a solution is not allowed by GR and also by CSG, because in this latter approach space-time is curved outside the string, whose scalar curvature depends on $f(x)$. These arguments justify the search of other possible solutions to Eq.(14). For that we first transform the integral (14) into a differential equation, under a change of the integration variable to $u(\theta) = 1/r$:

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{b^2} \tag{23}$$

which yields after a new differentiation

$$\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2} + u\right) = 0 \tag{24}$$

A first solution to the above equation is $du/d\theta = 0$ *i. e.* a circular path with radius b, according to Eq.(23). The solutions to the remaining equation in Eq.(24) are $1/r = b^{-1} \cos\theta$ (straight lines, as before), $1/r = b^{-1} \sin\theta$ and $1/r = b^{-1} \exp[i\theta]$. These latter solutions are meaningless and they must be discarded. The first interpretation of a circular path is given by GR, which states that photons follow archs of circles around a cosmic string (open trajectories, since $0 \leq \theta \leq \theta_M$ defines a wedge). However, the circular path of photons emerges naturally in CSG, because Eq.(14) does not depend on $f(x)$ although it is derived from the metric (3). Otherwise, since Witten's equation holds only outside matter, then in the CSG approach we cannot associate the energy-momentum tensor of the string to a field equation inside matter and then derive the angular deficit produced by the string, as is done in GR. Finally, since photons are "attracted" by the string in the CSG model and once we cannot have photons in bound states (closed circular paths), then all we can assert is that in the CSG framework the path of light near a cosmic string is also an open circular trajectory.

As a conclusion, we see that in three space-time dimensions, CSG and GR state that outside matter a first physical solution to the field equation (2) is the metric (22), which is a trivially conformally transformed of a flat space-time metric, where $0 \leq \theta \leq \theta_M$. The term trivial here means that $f(x)$ in Eq.(3) is an arbitrary dimensionless constant, when we look for a result which agrees with GR. However, the form $\mathring{g}_{ij} = f(x)\eta_{ij}$ also yields a circular path for a light ray.

Otherwise, it is important to recall that GR is not a conformally invariant theory in (3+1)-space-time dimensions [6]. This is corroborated by the present paper, because if the form (3) were extended to four space-time dimensions, we would conclude that such a metric is not a solution of Einstein's equation outside matter. Notice that the metric (3), when written in (3+1)-dimensions, yields a vanishing Weyl tensor (conformally flat space-time). Moreover, the field equation in the (3+1)-dimensional case is no longer Eq.(2). It is also important to recall that according to Witten's work [1], gravitation is a gauge theory in three space-time dimensions, however this is not true in four space-time dimensions. This suggests that gauge invariance and conformal transformations should be better investigated in gravitational theories.

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